International Journal of Theoretical Physics, Vol. 42, No. 10, October 2003 (© 2003)

Resonances and Virtual Poles in Scattering Theory

H. Baumgärtel[1]

Received June 30, 2003

Sufficient criteria for the coincidence of resonances and nonreal virtual poles in scattering systems are presented. A Gelfand triplet is constructed such that eigenfunctionals of the extended Hamiltonian exist exactly for the resonances.

KEY WORDS: resonances; virtual poles; scattering theory; Gelfand triplets.

1. INTRODUCTION

In this section we collect basic assumptions and results on our central objects to fix our notation. Let $\mathcal{H}$ be a separable Hilbert space and H a self-adjoint operator on $\mathcal{H}$. The domain dom $H \subset H$ is dense in $\mathcal{H}$, its resolvent is denoted by $R(z) := (z - H)^{-1}$. Its spectrum is real, spec $H \subseteq \mathbb{R}$. We assume that there is only eigenvalue spectrum and absolutely continuous spectrum.

The spectral measure of H is denoted by $E(\cdot)$. Borel sets of $\mathbb{R}$ are denoted by Δ. Recall the spectral theorem $H = \int_{-\infty}^{\infty} \lambda E(d\lambda)$. P^{ac} denotes the absolutely continuous projection. The projection $\mathbb{I} - P^{\mathrm{ac}} = P_0$ is the projection onto the closed linear span of all eigenvectors of H. For $f \in P^{\mathrm{ac}}\mathcal{H}$ one has: $\Delta \rightarrow (f, E(\Delta) f)$ is absolutely continuous w.r.t. the Lebesgue measure. The scalar product (f, g) on $\mathcal{H}$ is assumed to be antilinear in f and linear in g.

We assume that spec $\{H{\restriction}P^{\mathrm{ac}}\mathcal{H}\} = [0, \infty)$ i.e., the absolutely continuous spectrum is nonegative, and it has homogeneous multiplicity. Under our assumptions the *spectral representation theorem* reads as follows: $H{\restriction}P^{\mathrm{ac}}\mathcal{H}$ is unitarily equivalent to the multiplication operator by λ on the Hilbert space $P^{\mathrm{ac}}\mathcal{H} \cong L^2([0, \infty), d\lambda, \mathcal{K})$, $d\lambda$ the Lebesgue measure, where $\mathcal{K}$ denotes a separable Hilbert space; dim $\mathcal{K}$ represents the multiplicity of the absolutely continuous spectrum. Recall that the elements $f \in P^{\mathrm{ac}}\mathcal{H}$ are given by $\mathcal{K}$-valued functions

$$\mathbb{R}_+ \ni \lambda \rightarrow \hat{f}(\lambda) \in \mathcal{K} : \int_0^{\infty} \| \hat{f}(\lambda)\|_{\mathcal{K}}^2 \, d\lambda < \infty.$$

[1] Mathematical Institute, University of Potsdam Am Neuen Palais 10, Postfach 601 553, D-14415 Potsdam, Germany; e-mail: baumg@rz.uni-potsdam.de.

One has $(f, g) = \int_0^\infty (\hat{f}(\lambda), \hat{g}(\lambda))_{\mathcal{K}} d\lambda$, $E(\Delta)$ is given by the multiplication operator with the characteristic function $\chi_\Delta(\cdot)$, of Δ, the vector Hf corresponds to the function $\lambda \to \lambda \hat{f}(\lambda)$ and $e^{itH} f$ to $\lambda \to e^{it\lambda} \hat{f}(\lambda)$. One has

$$(f, E(\Delta) f) = \int_\Delta (\hat{f}(\lambda), \hat{f}(\lambda))_{\mathcal{K}} \, d\lambda,$$

and

$$\frac{f, E(d\lambda) f)}{d\lambda} = (\hat{f}(\lambda), \hat{f}(\lambda))_{\mathcal{K}},$$

exists a.e. on $[0, \infty]$.

The *evaluation operator* D_λ at λ is defined (in the moment pure formally) by

$$D_\lambda : L^2([0, \infty), d\lambda, \mathcal{K}) \to \mathcal{K}, \quad D_\lambda \hat{f} := \hat{f}(\lambda).$$

Later we will discuss existence questions for D_λ on appropriate submanifolds.

Further we introduce a second self-adjoint operator H_0 on $\mathcal{H}$, called the "free Hamiltonian." Again we assume that H_0 has only eigenvalue spectrum and absolutely continuous spectrum $[0, \infty)$ with homogeneous multiplicity. It is assumed that there is only a finite number of eigenvalues μ of finite multiplicity, which are embedded, i.e., $\mu \in (0, \infty)$. Then the projection $\mathbb{I} - P_0^{\mathrm{ac}} = P_0$ is finite-dimensional, where again P_0^{ac} denotes the absolutely continuous projection of H_0 and P_0 is then the projection onto the linear span of all eigenvalues of H_0.

The spectral measure of H_0 is denoted by $E_0(\cdot)$ and the corresponding evaluation operator by D_λ^0.

We assume that H and H_0 are connected by a so-called *perturbation* V, which is not necessarily bounded. First we give a pure formal ansatz: For technical reasons we put

$$V := A^* C A, \quad C = C^*, \quad B := CA.$$

C is a bounded operator on an auxiliary Hilbert space $\mathcal{F}$, which is introduced to have a measure $\|\mathcal{C}\|$ for smallness of the perturbation. A is a closed operator form $\mathcal{H}$ to $\mathcal{F}$. We put formally

$$H := H_0 + V.$$

In Section 3 we present criteria on V such that H is equipped with the properties mentioned before. Note that H may have negative eigenvalues. We mention the wave operators (Möller operators), denoted by $W_\pm$:

$$W_\pm := \text{s-lim}_{t \to \pm\infty} e^{itH} e^{-itH_0} P_0^{\mathrm{ac}}.$$

The so-called *completeness property* of the wave operators reads

$$W_\pm(P_0^{\mathrm{ac}} \mathcal{H}) = P^{\mathrm{ac}} \mathcal{H}.$$

In this case the scattering operator $S := W_+^* W_-$ commutes with all projections $E_0(\Delta)$ of the spectral measure of H_0 and $S{\restriction}P_0^{\mathrm{ac}}\mathcal{H}$ is unitary. Then, w.r.t. the spectral representation of H_0, S acts as multiplication operator by a unitary operator function

$$\mathbb{R}_+ \ni \lambda \to \hat{S}(\lambda) \in \mathcal{L}(\mathcal{K}_0),$$

the so-called scattering matrix, where $\mathcal{K}_0$ denotes the multiplicity Hilbert space for H_0. That is the vector $Sf \in \mathcal{H}$ corresponds to the function $\lambda \to \hat{S}(\lambda)\hat{f}(\lambda)$. We put $T := S - \mathbb{I}$. Then the function $\lambda \to \hat{T}(\lambda) = \hat{S}(\lambda) - \mathbb{I}_{\mathcal{K}_0}$ is called the scattering amplitude.

2. SOME PURE ALGEBRAIC RELATIONS AND THE LIVŠIC-MATRIX

In the following example $\mathfrak{I}z > 0$. Then we have the relation

$$\mathbb{I} + B(z - H)^{-1}A^* = (\mathbb{I} - B(z - H_0)^{-1}A^*)^{-1}.$$

Next we define two operator functions:

$$\Phi_+(z) := BP_0^{\perp}(z - H_0)^{-1}P_0^{\perp}A^*,$$
$$\Gamma_+(z) := (\mathbf{1} - \Phi_+(z))^{-1}.$$

Then the relation

$$\mathbb{I} + B(z - H)^{-1}A^* = \Gamma_+(z) + \Gamma_+(z)BP_0(z - H)^{-1}P_0A^*\Gamma_+(z)$$

holds. The operator function

$$P_0(z - H)^{-1}P_0$$

is called the *partial resolvent*. Further we obtain the relation

$$A^*\Gamma_+(z)B = V + VP_0^{\perp}(z - H_1)^{-1}P_0^{\perp}V,$$

where

$$H_1 := H_0 + P_0^{\perp}VP_0^{\perp} + P_0VP_0.$$

The auxiliary Hamiltonian H_1 commutes with P_0. To get the Livšicmatrix first we define

$$L_+(z) := zP_0 - H_0P_0 - P_0A^*\Gamma_+(z)BP_0$$
$$= zP_0 - H_1P_0 - P_0VP_0^{\perp}(z - H_1)^{-1}P_0^{\perp}VP_0.$$

Then one obtains for the partial resolvent the expression

$$P_0(z - H)^{-1}P_0 = P_0\{L_+(z){\restriction}P_0\mathcal{H}\}^{-1}P_0.$$

The operator function $z \to L_+(z) \upharpoonright P_0\mathcal{H}$ is called the Livšic-matrix. Note that P_0 is finite-dimensional. Therefore it is a matrix-valued function.

If we start with the lower half plane $\mathbb{C}_-$ then the corresponding functions are denoted by $\Phi_-(z)$, $\Psi_-(z)$ etc.

3. RESONANCES AND VIRTUAL POLES

3.1. Existence of the Scattering Matrix

Here we collect assumptions on V that guarantee Theorem 1 below.

In the following the upper half plane $\mathbb{C}_+ := \{z : \Im z > 0\}$ is used as the original domain for the operator functions considered, i.e., in the beginning let $\Im z > 0$.

A1.1: $AP_0^\perp$ is H_0-smooth (this implies that $AP_0^\perp(z - H_0)^{-1}$ is bounded). This means

$$\sup_{\epsilon > 0} \int_{-\infty}^{\infty} \{\|AR_0(\lambda + i\epsilon)u\|^2 + \|AR_0(\lambda - i\epsilon)u\|^2\}\, d\lambda \leq C_u < \infty$$

for all $u \in P_0^\perp\mathcal{H}$. This implies that $AR_0(\cdot + i0)u \in H_+^2(\mathbb{R}, \mathcal{F})$ and $AR_0(\cdot - i0)u \in H_-^2(\mathbb{R}, \mathcal{F})$ are Hardy class functions.

A1.2: The function

$$\mathbb{C}_+ \ni z \to AP_0^\perp(z - H_0)^{-1} P_0^\perp A^* \tag{1}$$

is holomorphically continuable across $\mathbb{R}_+ = (0, \infty)$ and defines a holomorphic operator function on $\mathbb{C}_{<0} := \mathbb{C}\backslash(-\infty, 0]$. This implies that $AP_0^\perp R_0(\lambda + i0)P_0^\perp A^*$ is bounded for every $\lambda > 0$.

(Note that (1) is a priori holomorphic not only on $\mathbb{C}_+$ but holomorphic in $\mathbb{C}_{>0} := \mathbb{C}\backslash[0.\infty)$. Therefore the lower half plane of $\mathbb{C}_{<0}$ is called the *second sheet* of (1), whereas the lower half plane of $\mathbb{C}_{>0}$ is the first sheet.)

A2: $\sup_{\lambda>0}\|AP_0^\perp R_0(\lambda + i0)P_0^\perp A^*\| =: a < \infty$.
A3: $\|\mathcal{C}\| < \frac{1}{a}$ (smallness condition).

Then $\sup_{\lambda>0}\|\Phi_+(\lambda + i0)\| < 1$ follows and $\Gamma_+(\lambda + i0) = (\mathbb{I} - \Phi_+(\lambda + i0))^{-1}$ is holomorphic on $\mathbb{R}_+ = \{\lambda : \lambda > 0\}$. Then, according to a theorem of Gochberg/Krein (see Gochberg and Krein, 1957), one obtains that $z \to \Gamma_+(z)$ is meromorphic on $\mathbb{C}_{<0}$. Using the relations of Section 2 this implies that also

$$z \to \Psi_+(z) := B(z - H)^{-1}A^*$$

is meromorphic on $\mathbb{C}_{<0}$. The poles of Ψ_+ are called *virtual poles*. Since Ψ_+ is holomorphic on $\mathbb{C}_+$, for virtual poles ζ one has necessarily $\Im\zeta \leq 0$. If ζ is real then ζ is an eigenvalue of H (this is used in the proof of Theorem 3).

Theorem 1. *Assume conditions A1–A3. Then: H is self-adjoint, the wave operators $W_{\pm}(H, H_0)$ exist and are complete. H satisfies the conditions mentioned in Section 1.*

Therefore the scattering matrix $\hat{S}(\lambda)$ exists as a unitary operator on $\mathcal{K}_0$ a.e on $\mathbb{R}_+$.

3.2. Analytic Continuation of the Scattering Matrix

In order to get analytic continuation for the scattering matrix, we need a stronger assumption.

A4: $F_A(\lambda) := D_\lambda^0 A^*$ is a bounded operator from $\mathcal{F}$ to $\mathcal{K}_0$ for all $\lambda > 0$ and this function $F_A(\cdot)$ is holomorphic continuable to $\mathbb{C}_{<0}$.

A4 is a strengthening of A1, i.e., A4 implies A1. Recall (first formally) that

$$\frac{AP_0^\perp E_0(d\lambda)P_0^\perp A^*}{d\lambda} = \frac{1}{2i\pi}(AP_0^\perp R_0(\lambda - i0)P_0^\perp A^* - AP_0^\perp R_0(\lambda + i0)P_0^\perp A^*)$$

or

$$AP_0^\perp R_0(\lambda + i0)P_0^\perp A^* = AP_0^\perp R_0(\lambda - i0)P_0^\perp A^* - 2i\pi \frac{AP_0^\perp E_0(d\lambda)P_0^\perp A^*}{d\lambda}.$$

The left-hand side is, according to A1, holomorphic continuable on $\mathbb{C}_{<0}$. Therefore, the first term on the right-hand side is also holomorphic continuable on $\mathbb{C}_{<0}$ (starting with the lower half plane of the first sheet).

Hence also

$$\frac{AP_0^\perp E_0(d\lambda P_0^\perp A^*}{d\lambda}$$

is holomorphic continuable on $\mathbb{C}_{<0}$. For $u, v \in \mathcal{F}$ we have

$$\left(u, \frac{AP_0^\perp E_0(d\lambda)P_0^\perp A^* v}{d\lambda}\right) = \frac{(P_0^\perp A^* u, E_0(d\lambda)P_0^\perp A^* v)}{d\lambda} = \left(D_\lambda^0 P_0^\perp A^* u, D_\lambda^0 P_0^\perp A^* v\right),$$

i.e., A1 implies and is equivalent to the statement that

$$\phi_{u,v}(\lambda) := \left(D_\lambda^0 P_0^\perp A^* u, D_\lambda^0 P_0^\perp A^* v\right)$$

is holomorphic on $\mathbb{C}_{<0}$ for all $u, v \in \mathcal{F}$. Therefore, if A4 is satisfied then also the last statement, i.e., A1 is true.

If $FA(\cdot)$ is bounded then there is a well-known expression for the scattering amplitude $\hat{T}(\cdot)$ (see, e.g., Baumgärtel and Wollenberg [1983, p. 393]):

$$\hat{T}(\lambda) = -2i\pi F_A(\lambda)(C + CAR(\lambda + i0)A^* C)F_A(\lambda)^*.$$

From this formula one obtains immediately

Theorem 2. *Assume the conditions A2–A4. Then: $\hat{T}(\cdot)$ is meromorphic on $\mathbb{C}_{<0}$.*

The poles of $\hat{T}$ are called *resonances*.

3.3. Coincidence of Resonances and Nonreal Virtual Poles

Recall that the functions Ψ_+ and $\hat{T}$ are both meromorphic on $\mathbb{C}_{<0}$.

Theorem 3. *The poles of $\hat{T}$ and the nonreal poles of Ψ_+ coincide.*

Proof: It is sufficient to prove that a nonreal pole ζ of Ψ_+ is a pole of $\hat{T}$ and a real pole of Ψ_+ is a holomorphic point for $\hat{T}$.

1. We have

$$CF_A(\lambda)^*\hat{T}(\lambda)F_A(\lambda) = -2i\pi CF_A(\lambda)^*F_A(\lambda) \times (C + \Psi_+(\lambda)C)F_A(\lambda)^*F_A(\lambda).$$

It is sufficient to show that ζ is a pole of the left-hand side. Note that

$$CF_A(\lambda)^*F_A(\lambda) = \frac{1}{2i\pi}CAP_0^\perp(R_0(\lambda - i0) - R_0(\lambda + i0))P_0^\perp A^* = \frac{1}{2i\pi}(\Phi_-(\lambda) - \Phi_+(\lambda)).$$

This gives

$$CF_A(\overline{z})^*\hat{T}(z)F_A(z) = -\frac{1}{2i\pi}(\Phi_-(z) - \Phi_+(z))(\mathbb{I} + \Psi_+(z))(\Phi_-(z) - \Phi_+(z)).$$

Recall $\mathbb{I} + \Psi_+(z) = (\mathbb{I} - \Phi_+))^{-1}$. This implies

$$\Phi_+(z)\Psi_+(z) = \Psi_+(z)\Phi_+(z) = \Psi_+(z) - \Phi_+(z).$$

Let ζ be a pole of order m, i.e.,

$$\Psi_+(z) = (z - \zeta)^{-m}D_\zeta + C(z), \qquad D_\zeta \neq 0, \quad m \geq 1.$$

Then

$$\lim_{z\to\zeta}(z - \zeta)^m(\Phi_-(z) - \Phi_+(z))(\mathbb{I} + \Psi_+(z))(\Phi_-(z) - \Phi_+(z))$$
$$= (\Phi_-(\zeta) - \Phi_+(\zeta))D_\zeta(\Phi_-(\zeta) - \Phi_+(\zeta)) = (\mathbb{I} - \Phi_-(\zeta))D_\zeta(\mathbb{I} - \Phi_-(\zeta))$$

because of $D_\zeta = \Phi_+(\zeta)D_\zeta = D_\zeta\Phi_+(\zeta)$. But $\mathbb{I} - \Phi_-(\zeta) = \mathbb{I} - CAR_0(\zeta)A^*$ and $(\mathbb{I} - \Phi_-(\zeta))^{-1} = \mathbb{I} + CAR(\zeta)A^*$. Hence

$$(\mathbb{I} - \Phi_-(\zeta)D_\zeta(\mathbb{I} - \Phi_-(\zeta)) \neq 0$$

follows.

2. A real pole ξ of Ψ_+ is a holomorphic point for $\hat{T}$: First ξ is necessarily a simple pole and an eigenvalue of H. The residuum of Ψ_+ at ξ is given by $D := BQA^*$, where Q is the eigenprojection of ξ w.r.t. H, i.e., we have

$$\Psi_\pm(z) = \frac{BQA^*}{z - \zeta} + BQ^\perp(z - H)^{-1}Q^\perp A^*.$$

Note that the residuum is the same for both functions Ψ_- and Ψ_+. Further we use again $(\mathbb{I} + \Psi_\pm(z)) = (\mathbb{I} - \Phi_\pm(z))^{-1} \cdot \xi$ is a holomorphic point for $\Phi_-(z) - \Phi_+(z)$. We have to check the expression

$$(\Phi_-(z) - \Phi_+(z)\left(\frac{D}{z - \xi} + C(z)\right)(\Phi_-(z) - \Phi_+(z)).$$

Using $0 = D(\Phi_-(\xi) - \Phi_+(\xi)) = (\Phi_-(\xi) - \Phi_+(\xi))D$, we obtain that ξ is a holomorphic point for $\hat{T}$. □

3.4. A Special Case: There Are No Embedded Eigenvalues (of H_0)

This means that $P_0 = 0$. In this case we have $\mathbb{I} + \Psi_+(z) = \Gamma_+(z)$ and $F_A(\lambda) = D^0_\lambda A^*$. Therefore, Ψ_+ and Ψ_- are both holomorphic for $\lambda > 0$ and

$$\frac{AE(d\lambda)A^*}{d\lambda}$$

is meromorphic on $\mathbb{C}_{<0}$.

As an illustration we consider a special case: Let $C = \mathbb{I}$, H cyclic with generating unit vector $e \in P^{\mathrm{ac}}\mathcal{H}$. In other words, we assume the multiplicity to be 1. Let $e = A^*e_0$. Then

$$(e, E(\Delta)\, e) = (e_0, AE(\Delta)A^*\, e_0)$$

and

$$\rho(\lambda) := \frac{(e_0, AE(d\lambda)A^*\, e_0)}{d\lambda}$$

is holomorphic on $\mathbb{R}_+$ and meromorphic on $\mathbb{C}_{<0} \cdot \rho(\cdot)$ is called the *spectral density*.

Now let f, g be vectors from $P^{\mathrm{ac}}\mathcal{H}$, generated by the functions ϕ, Ψ, respectively, i.e., $f = \phi(H)e, g = \psi(H)e$. Then $f = W_+ f_0, g = W_- g_0$ where $f_0, g_0 \in \mathcal{H}$ and

$$(f_0, S_{g0}) = \int_0^\infty \overline{\phi(\lambda)}\psi(\lambda)\rho(\lambda)\, d\lambda.$$

Now choose ϕ and ψ as restrictions to $\lambda > 0$ of functions $\phi \in H^2_+(\mathbb{R})$, $\psi \in H^2_-(\mathbb{R})$. Then

$$F(\lambda) := \overline{\phi(\lambda)}\psi(\lambda)\rho(\lambda)$$

is meromorphic on $\mathbb{C}_-$, the lower half plane, where possible poles are only due to $\rho(\cdot)$. Then $\int_{-R}^{+R} + \int_C = -2i\pi \text{Res}\,\{F(z)\}$ where C denotes the negatively oriented semicircel from $+R$ to $-R$ in the lower half plane and where Res means the sum of all residua inside the corresponding semidisc. If $\lim_{R\to\infty}\int_C = 0$ then one has

$$(f_0, S_{g0}) = \int_0^{-\infty} \overline{\phi(\lambda)}\psi(\lambda)\rho(\lambda),\, d\lambda - 2i\pi \text{Res}_{\mathfrak{I}z<0}\{F(z)\}.$$

4. EIGENFUNCTIONALS FOR RESONANCES

In this section we consider the special case that there are embedded eigenvalues of H_0, i.e. we assume

B1: $P_0 > 0$,
B2: $z \to \Gamma_+(z)$ is holomorphic on $\mathbb{C}_{<0}$, and
B3: A, B hence V are bounded.

This case is a counterpart to Section 3. B3 is assumed to avoid technical domain discussions. Then we obtain immediately from Section 2.

Proposition 1. *Assume additionally B1–B3. Then: The virtual poles are exactly the poles of the partial resolvent, i.e., they are the zeros of the determinant of the Livšic-matrix*

$$\det\{L_+(z) \restriction P_0\mathcal{H}\}.$$

In this case one can introduce appropriate eigenfunctionals for H exactly for the resonances, but not for other (nonreal) complex numbers.

4.1. Construction of the Gelfand Triplet

The idea is to use the spectral representation of H_1. Recall that B1–B3 together with the results of Section 3 imply that also the wave operators $W_\pm(H_1, H_0)$ exist and that they are complete, which implies that $H_1 \restriction P_0^{\text{ac}}\mathcal{H}$ and $H_0 \restriction P_0^{\text{ac}}\mathcal{H}$ are unitarily equivalent. Therefore, the absolutely continuous spectrum of H_1 is $[0, \infty]$ and it has homogeneous multiplicity. We denote the corresponding multiplicity subspace by $\mathcal{K}_1$ and the evaluation operator by D^1_λ.

Now we define a linear manifold $\Phi \subset P_0^{\mathrm{ac}}\mathcal{H}$. A vector $f \in P_0^{\mathrm{ac}}\mathcal{H}$ is an element of Φ iff

$$\lambda \to \left(D_\lambda^1 f, D_\lambda^1(P_0^\perp V h)\right)_{\mathcal{K}_1},$$

is holomorphic on $\mathbb{C}_{<0}$ for all $h \in P_0\mathcal{H}$.

Then from Section 2 we obtain that

$$P_0^\perp V P_0 \mathcal{H} \subset \Phi.$$

Φ is dense in $P_0^{\mathrm{ac}}\mathcal{H}$ (see Baumgärtel, 1976). We omit the explicit construction of a suitable locally convex topology in Φ such that Φ is complete and continuously embedded in $P_0^{\mathrm{ac}}\mathcal{H}$.

Since H_1 is unbounded in general, one has to introduce additionally the linear manifold $\Phi_1 : \mathrm{dom}(H_1 \restriction P_0^{\mathrm{ac}}) \cap \Phi$, equipped with a slightly changed topology (see Baumgärtel, 1976). Then H_1 is a continuous linear operator from Φ_1 into Φ, since together with $(D_\lambda^1 f, D_\lambda^1(P_0^\perp V h))_{\mathcal{K}_1}$ also $D_\lambda^1 H_1 f, D_\lambda^1(P_0^\perp V h))_{\mathcal{K}_1} = \lambda(D_\lambda^1 f, D_\lambda^1(P_0^\perp V h))_{\mathcal{K}_1}$ is holomorphic on $\mathbb{C}_{<0}$.

As the fundamental manifold in $\mathcal{H}$ we choose $\mathcal{D} := \Phi \oplus P_0\mathcal{H}$. We equip $\mathcal{D}$ with the product topology of Φ and $P_0\mathcal{H}$. Then $\mathcal{D}$ is continuously embedded in $\mathcal{H}$. We have $\mathcal{D}^* = \Phi^* \times P_0\mathcal{H}$, i.e., the antilinear forms from $\mathcal{D}^*$ are pairs (ϕ^*, h_0) with $\phi^* \in \Phi^*$ and $h_0 \in P_0\mathcal{H}$, such that

$$\langle(\phi, x) \mid (\phi^*, h_0)\rangle := \langle \phi \mid \phi^* \rangle + (x, h_0), \quad \phi \in \Phi, \ x \in P_0\mathcal{H}.$$

The Hilbert space $\mathcal{H}$ is canonically embedded into $\mathcal{D}^*$, that is we obtain the Gelfand triple

$$\mathcal{D} \subset \mathcal{H} \subset \mathcal{D}^*. \tag{2}$$

Correspondingly we introduce $\mathcal{D}_1 := \Phi_1 \oplus P_0\mathcal{H}$. Then H is a continuous linear operator from $\mathcal{D}_1$ into $\mathcal{D}$ because for any $x \in \mathcal{H}$ one has $(P_0 V P_0^\perp + P_0^\perp V P_0)x \in \mathcal{D})$. The extension H^* of H w.r.t. (2) (the so-called Gelfand triplet adjoint) is defined by

$$\langle(\phi, x) \,|\, H^*(\phi^*, h_0)\rangle := \langle H(\phi, x) \,|\, (\phi^*, h_0)\rangle.$$

4.2. Eigenfunctionals for H^*

The eigenvalue equation for H^* reads

$$H^*(\phi^*, h_0) = \zeta(\phi^*, h_0), \quad \zeta \in \mathbb{C}_{<0}.$$

Theorem 4. *The nonreal complex number $\zeta \in \mathbb{C}_{<0}$ is an eigenvalue of H^* iff ζ is a resonance. In this case the eigenspace for ζ (i.e. the linear span of all* *eigenvectors) is given by*

$$\ker\{L_+(\zeta) \restriction P_0\mathcal{H}\} \subset P_0\mathcal{H}.$$

Proof: The eigenvalue equation can be split into two separate equations:

$$\phi = 0 : (\overline{\zeta} x - P_0 H_1 P_0 x, h_0) = \langle P_0^\perp V_x \mid \phi^* \rangle, \tag{3}$$

$$x = 0 : \langle(\overline{\zeta} - H_1)\phi \mid \phi^* \rangle = (\phi, P_0^\perp V h_0). \tag{4}$$

For $\mathfrak{J}\zeta > 0$ the solution of (4) is given by

$$\langle \phi \mid \phi_\zeta^* \rangle := (\phi, (\zeta - H_1)^{-1} P_0^\perp V h_0.$$

Since $(D_\lambda^1 \phi, D_\lambda^1 (P_0^\perp V h_0))_{\mathcal{K}_1}$ is holomorphic on $\mathbb{C}_{<0}$ the antilinear form ϕ_ζ^* is holomorphic on $\mathbb{C}_{<0}$. Inserting this solution into (3), we get $(x, (\zeta - H_1)h_0) = \langle P_0^\perp V x \mid \phi_\zeta^* \rangle$, i.e.,

$$D_x(\zeta) := (x, (\zeta - H_1)h_0) - \langle P_0^\perp V x \mid \phi_\zeta^* \rangle$$

should vanish for all $x \in P_0\mathcal{H}$. First let $\mathfrak{J}z > 0$. Then

$$D_x(z) = (x, \{z - H_1 - P_0 V P_0^\perp (z - H_1)^{-1} P_0^\perp V\} h_0) = (x, L_+(z) h_0),$$

but this function is even holomorphic on $\mathbb{C}_{<0} \cdot D_x(z) = 0$ for all $x \in P_0\mathcal{H}$ means simply $L_+(z)h_0 = 0$, i.e., a solution $h_0 \in P_0\mathcal{H}$, $h_0 \neq 0$, for a parameter z with $\mathfrak{J}z < 0$ exists iff $z := \zeta$ is a resonance. The eigen(anti-)linear forms for a resonance ζ are given by

$$(\phi_{\zeta, h_0}, h_0), \quad h_0 \in \ker\{L_+(\zeta) \restriction P_0\mathcal{H}\},$$

where the antilinear form ϕ_{z,h_0}^* for $\mathfrak{J}z > 0$ is given by

$$\langle \phi \mid \phi_{z,h_0}^* \rangle := (\phi, (z - H_1)^{-1} P_0^\perp V h_0). \qquad \square$$

REFERENCES

Baumgärtel, H. (1976). Resonances of Perturbed selfadjoint operators and their Eigenfunctionals. *Mathematische Nachrichten* **75**, 133–151.

Baumgärtel, H. and Wollenberg, M. (1983). *Mathematical Scattering Theory*, Birkhäuser, Boston, MA.

Gochberg, I. C. and Krein, M. G. (1957). The basic propositions on defect numbers, root numbers and indices of linear operators. *Uspekhi Matematicheskikh Nauk* **12**, 43–118. (In Russian)